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Chaotic dynamics in periodically forced asymmetric ordinary differential equations ☆

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ABSTRACT

Using a topological approach, we prove the existence of infinitely many periodic solutions and the presence of chaotic dynamics for the periodically forced second order ODE $u'' + bu^+ - au^- = p(t)$. The choice of the equation is motivated by the studies about the Dancer–Fučik spectrum and the Lazer–McKenna suspension bridge model.

1. Introduction and statement of the main results

In a recent paper [29] we have proved the existence of chaotic dynamics associated to the second order ODE

$$u'' + ku^+ = p_{r,s}(t) \quad (k > 0), \tag{1}$$

for a suitable choice of a T-periodic forcing term $p_{r,s}(\cdot)$ which changes its sign on the time interval $[0, T]$. Such an equation represents a simplified version of the Lazer–McKenna model for the oscillations of suspension bridges. In [18–20] the authors considered a more general periodically forced equation of the form

$$u'' + bu^+ - au^- = p(t) \tag{2}$$

as well as its "damped" counterpart

$$u'' + cu' + bu^+ - au^- = p(t) \quad (c > 0), \tag{3}$$

with

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$$b > 0, \quad 0 \leqslant a \neq b.$$

Special emphasis, from the point of view of the applications, was also put in [18] to positive forcing terms like

$$p(t) = L\big(1 + \varepsilon h(t)\big) \quad (L, \varepsilon > 0).$$

It is the aim of the present article to show that the topological approach and the arguments employed in [29] can be adapted and applied to these more general equations, by showing that, for a suitable choice of a periodic forcing term $p_{r,s}(\cdot)$, both

$$u'' + bu^+ - au^- = p_{r,s}(t) \tag{4}$$

and Eq. (3) for

$$p(t) = p_{r,s}(t) + q(t), \quad \text{with } |q|_{L^1([0,T])} + |c| \text{ small},$$

have chaotic solutions. For sake of simplicity, we will discuss in detail only the case $a, b > 0$ (with $a \neq b$). The analysis of the case $a = 0 < b$, which corresponds to Eq. (1) already considered in [29] (but with different conditions on the forcing term) turns out to be a minor variant of the main result of this paper, and therefore it will be only commented in Remark 2.1 at the end of the proof. The conditions on the forcing term $p_{r,s}(\cdot)$ considered in this paper are more general than those in [29] since here we do not require that $p_{r,s}(t)$ changes its sign in a period, as we did in [29] (motivated also from [23]).

The explicit form of $p_{r,s}(t)$ is that of a rectangular wave which oscillates back and forth between the constant values r and s (see (8) for the precise definition of $p_{r,s}$). Throughout the paper we will consider only the case $s < r$, with $rs \neq 0$. The argument involving the case $s > r$ is exactly the same (with obvious modifications) and therefore is omitted. We choose a piecewise constant forcing term in order to simplify the technicalities in the proofs. However, we remark that the consideration of this kind of forcing terms is relevant in control theory and, moreover, it has found its own interest in different areas of ODEs, as the analysis of some mechanical or electrical circuits models [6], in mathematical biology [14,26] and in the study of the Littlewood boundedness problem [21,22,37]. We also recall the pioneering work of Osipov and Pliss [28] where the authors discovered the presence of “stochastic motions” for a second order Duffing equation, under the effect of a piecewise constant periodic forcing term. The equation considered in [28], after a change of variables, reads as

$$y'' + y' + b\big(y + y^2\big) = af(\varepsilon t), \qquad f(\theta) = \operatorname{sign}\big(\sin(\pi\theta/\omega)\big),$$

with given constants $a, b > 0$ and $\varepsilon > 0$ small. In our opinion, it seems interesting to remark that, although the phase portraits associated to Osipov and Pliss’s equation and to our Eq. (4) are completely different, nevertheless in both cases the chaotic dynamics are obtained for large periods of the forcing term (see also Section 3.3 for more comments concerning such kind of “slow chaos”).

Prior to presenting our results for Eq. (4) we recall, for the reader’s convenience, the topological tools and the concept of chaos which are used in this paper. Borrowing notation and terminology from [29], we start with the following definitions. By *path* γ in a metric space X we mean a continuous mapping $\gamma : \mathbb{R} \supseteq [t_0, t_1] \to X$ and we set $\bar{\gamma} := \gamma([t_0, t_1])$. Without loss of generality we will usually take $[t_0, t_1] = [0, 1]$. By a *sub-path* σ of γ we mean just the restriction of γ to a compact subinterval of its domain. An *arc* is the homeomorphic image of the compact interval $[0, 1]$. We define an *oriented rectangle* in X, as a pair

$$\widetilde{\mathcal{R}} := \big(\mathcal{R}, \mathcal{R}^-\big),$$

where $\mathcal{R} \subseteq X$ is homeomorphic to the unit square $[0, 1]^2$ by a homeomorphism $\eta : [0, 1]^2 \to \eta([0, 1]^2) = \mathcal{R}$ and

$$\mathcal{R}^- := \mathcal{R}^-_1 \cup \mathcal{R}^-_2$$

is the union of two disjoint compact arcs $\mathcal{R}^-_1, \mathcal{R}^-_2 \subseteq \partial\mathcal{R} := \eta(\partial[0, 1]^2)$ (which are called the “components” or “sides” of $\mathcal{R}^-$). We also denote by $\mathcal{R}^+$ the closure of $\partial\mathcal{R} \setminus (\mathcal{R}^-_1 \cup \mathcal{R}^-_2)$ which is the union of two disjoint compact arcs $\mathcal{R}^+_1$ and $\mathcal{R}^+_2$. Given an oriented rectangle $(\mathcal{R}, \mathcal{R}^-)$ it is always possible to prove the existence of a homeomorphism $h : [0, 1]^2 \to \mathcal{R} = h([0, 1]^2) \subseteq X$ such that $h(\{0\} \times [0, 1]) = \mathcal{R}^-_1$, $h(\{1\} \times [0, 1]) = \mathcal{R}^-_2$ and $h([0, 1] \times \{0\}) \cup h([0, 1] \times \{1\}) = \mathcal{R}^+$.

Suppose now that $\phi : X \supseteq D_\phi \to X$ is a map (not necessarily continuous everywhere on its domain D_ϕ) and let $\widetilde{\mathcal{A}} := (\mathcal{A}, \mathcal{A}^-)$ and $\widetilde{\mathcal{B}} := (\mathcal{B}, \mathcal{B}^-)$ be oriented rectangles. Let also $\mathcal{H} \subseteq \mathcal{A} \cap D_\phi$ be a compact set.

Definition 1.1. We say that $(\mathcal{H}, \phi)$ *stretches* $\widetilde{\mathcal{A}}$ *to* $\widetilde{\mathcal{B}}$ *along the paths* and write

$$(\mathcal{H}, \phi) : \widetilde{\mathcal{A}} \;{\Leftarrow\!\!\!\!\Rightarrow}\; \widetilde{\mathcal{B}},$$

if the following conditions hold:

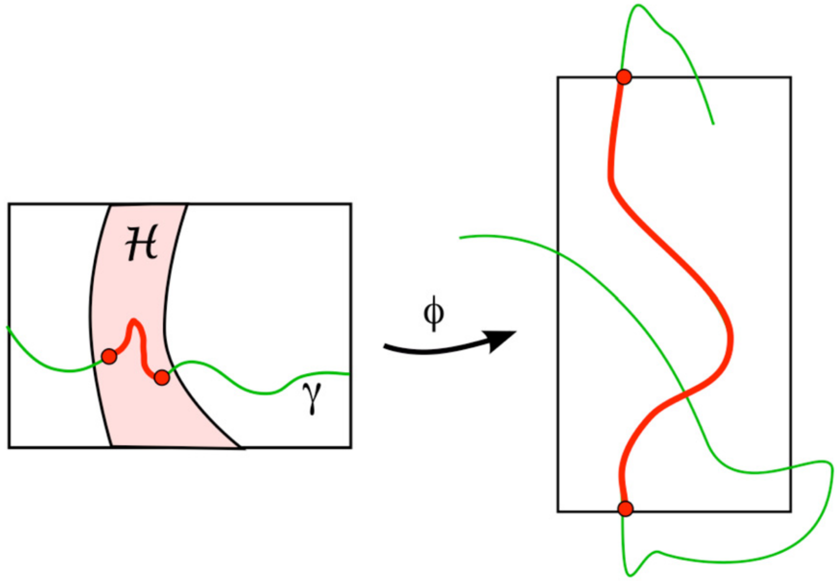

Fig. 1. A visualization of the stretching definition. The set $\mathcal{A}$ is the rectangle on the left, with $\mathcal{A}^-$ the union of its vertical edges. The set $\mathcal{B}$ is the rectangle on the right, with $\mathcal{B}^-$ the union of its horizontal edges.

- ϕ is continuous on $\mathcal{H}$;
- for every path $\gamma : [t_0, t_1] \to \mathcal{A}$ such that $\gamma(t_0) \in \mathcal{A}_1^-$ and $\gamma(t_1) \in \mathcal{A}_2^-$ (or $\gamma(t_0) \in \mathcal{A}_2^-$ and $\gamma(t_1) \in \mathcal{A}_1^-$), there exists a subinterval $[t', t''] \subseteq [t_0, t_1]$ such that

$$\gamma(t) \in \mathcal{H}, \qquad \phi\big(\gamma(t)\big) \in \mathcal{B}, \quad \forall t \in [t', t'']$$

and, moreover, $\phi(\gamma(t'))$ and $\phi(\gamma(t''))$ belong to different components of $\mathcal{B}^-$.

In the special case in which $\mathcal{H} = \mathcal{A}$, we simply write

$$\phi : \widetilde{\mathcal{A}} \,{\leftrightharpoons\!\!\!\rightarrow}\, \widetilde{\mathcal{B}}.$$

The stretching property (see Fig. 1.), which is related to the theory of *topological horseshoes* [5,12,15,16,36], provides us the main tool for the proof of chaotic dynamics in virtue of the following theorem.

Theorem 1.1. *Let $\psi_r : X \supseteq D_{\psi_r} \to X$ and $\psi_s : X \supseteq D_{\psi_s} \to X$ be given maps. Let $\widetilde{\mathcal{M}} := (\mathcal{M}, \mathcal{M}^-)$ and $\widetilde{\mathcal{N}} := (\mathcal{N}, \mathcal{N}^-)$ be oriented rectangles in X with $\mathcal{M} \subseteq D_{\psi_r}$ and $\mathcal{N} \subseteq D_{\psi_s}$. Suppose that the following conditions are satisfied:*

(H_r) *There exists $m \geqslant 2$ pairwise disjoint compact sets $\mathcal{K}_1, \ldots, \mathcal{K}_m \subseteq \mathcal{M}$ such that*

$$(\mathcal{K}_i, \psi_r) : \widetilde{\mathcal{M}} \,{\leftrightharpoons\!\!\!\rightarrow}\, \widetilde{\mathcal{N}}, \quad \forall i = 1, \ldots, m;$$

(H_s) $\psi_s : \widetilde{\mathcal{N}} \,{\leftrightharpoons\!\!\!\rightarrow}\, \widetilde{\mathcal{M}}$.

Then the map $\psi := \psi_s \circ \psi_r$ induces chaotic dynamics on m symbols in the set

$$\mathcal{K} := \bigcup_{i=1}^{m} \mathcal{K}_i.$$

Moreover, for each sequence of m symbols $\mathbf{s} = (s_n)_n \in \{1, \ldots, m\}^{\mathbb{N}}$, there exists a compact connected set $\mathcal{C}_\mathbf{s} \subseteq \mathcal{K}_{s_0}$ with

$$\mathcal{C}_\mathbf{s} \cap \mathcal{M}_1^+ \neq \emptyset, \qquad \mathcal{C}_\mathbf{s} \cap \mathcal{M}_2^+ \neq \emptyset$$

and such that, for every $w \in \mathcal{C}_\mathbf{s}$ there exists a sequence $(y_n)_n$ with $y_0 = w$ and

$$y_n \in \mathcal{K}_{s_n}, \qquad \psi(y_n) = y_{n+1}, \quad \forall n \geqslant 0.$$

See [29] for a proof of Theorem 1.1 and [30] for some remarks and extensions. In the applications of Theorem 1.1 to the ODE models considered in this paper, we will have $X = \mathbb{R}^2$ and the maps ψ_r, ψ_s will be the Poincaré maps associated to some planar systems.

At last, we explain in which sense we speak of *chaotic dynamics* in Theorem 1.1.

Definition 1.2. Let X be a metric space, let $\psi : X \supseteq D_\psi \to X$ be a map and $\mathcal{D} \subseteq D_\psi$ a nonempty set. Assume also that $m \geqslant 2$ is an integer. We say that ψ *induces chaotic dynamics on m symbols in the set $\mathcal{D}$* if there exist m nonempty pairwise disjoint compact sets

$$\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_m \subseteq \mathcal{D},$$

such that for every two-sided sequence of m symbols

$$(s_i)_{i\in\mathbb{Z}} \in \Sigma_m := \{1, \ldots, m\}^{\mathbb{Z}},$$

there exists a corresponding sequence $(w_i)_{i\in\mathbb{Z}} \in \mathcal{D}^{\mathbb{Z}}$ with

$$w_i \in \mathcal{K}_{s_i} \quad \text{and} \quad w_{i+1} = \psi(w_i), \quad \forall i \in \mathbb{Z} \tag{5}$$

and, whenever $(s_i)_{i\in\mathbb{Z}}$ is a k-periodic sequence (that is $s_{i+k} = s_i$, $\forall i \in \mathbb{Z}$) for some $k \geqslant 1$, there exists a k-periodic sequence $(w_i)_{i\in\mathbb{Z}} \in \mathcal{D}^{\mathbb{Z}}$ satisfying (5).

This definition corresponds to the concept of chaos in the sense of coin-tossing (taken from [17]) improved with the information about the existence of periodic points associated to periodic sequences of symbols. Definition 1.2 agrees with other ones considered in the literature about chaotic dynamics for ODEs with periodic coefficients (see [7,31,35], where ψ is the Poincaré map associated to a differential system). We notice that if the map ψ fulfills Definition 1.2 and is also *continuous and injective on*

$$\mathcal{K} := \bigcup_{i=1}^{m} \mathcal{K}_i \subseteq \mathcal{D}$$

(like in the case of a Poincaré map), then there exists a nonempty compact set

$$\Lambda \subseteq \mathcal{K}$$

which is invariant for ψ (i.e., $\psi(\Lambda) = \Lambda$) and such that $\psi|_\Lambda$ is semiconjugate to the two-sided Bernoulli shift σ on m symbols

$$\sigma : \Sigma_m \to \Sigma_m, \qquad \sigma\big((s_i)_{i\in\mathbb{Z}}\big) = (s_{i+1})_{i\in\mathbb{Z}},$$

according to the commutative diagram

$$\begin{tikzcd}
\Lambda \arrow[r, "\psi"] \arrow[d, "g"'] & \Lambda \arrow[d, "g"] \\
\Sigma_m \arrow[r, "\sigma"'] & \Sigma_m
\end{tikzcd}$$

where g is a continuous and surjective function. Moreover, the subset of Λ consisting of the periodic points of ψ is dense in Λ and the counterimage (by the semiconjugacy) of any periodic sequence in Σ_m contains a periodic point of ψ (see [30] for the details). Note that the semiconjugation to the Bernoulli shift is one of the typical requirements for chaotic dynamics as it implies a positive topological entropy for the map $\psi|_\Lambda$ [5].

After the introduction of the main topological tools, we are ready now to start with an application to the Lazer–McKenna equation.

Let us consider the periodically forced second order scalar differential equation (4), where

$$a > 0, \qquad b > 0, \qquad a \neq b. \tag{6}$$

The forcing term $p_{r,s} : \mathbb{R} \to \mathbb{R}$ is a periodic function of period

$$T = T_{r,s} := \tau_r + \tau_s \quad (\text{with } \tau_r > 0 \text{ and } \tau_s > 0), \tag{7}$$

defined on $[0, T[$ by

$$p_{r,s}(t) := \begin{cases} r, & \text{for } 0 \leqslant t < \tau_r, \\ s, & \text{for } \tau_r \leqslant t < \tau_r + \tau_s, \end{cases} \tag{8}$$

with

$$r \neq 0, \qquad s \neq 0, \qquad r > s, \tag{9}$$

and then extended to the real line by T-periodicity. As usual, u^+ and u^- are defined as

$$u^+ := \max\{u, 0\} = \frac{u + |u|}{2}, \qquad u^- := \max\{-u, 0\} = \frac{|u| - u}{2},$$

with $u = u^+ - u^-$ and $|u| = u^+ + u^-$.

In order to present our results for Eq. (4), we analyze the phase-portrait associated to the planar autonomous system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -(bx^+ - ax^-) + \mu \end{cases} \tag{E_μ}$$

depending by the parameter $\mu \in \mathbb{R} \setminus \{0\}$ (with $\mu \in \{r, s\}$). Clearly, (E_μ) is a conservative system of the form

$$\dot{x} = y, \qquad \dot{y} = -g(x),$$

with $g = g_\mu : \mathbb{R} \to \mathbb{R},\ g(x) = bx^+ - ax^- - \mu$, a Lipschitz function depending on the parameter μ. The corresponding orbits in the phase-plane lie on the level lines of the energy

$$\mathcal{E}^\mu(x, y) := \frac{1}{2} y^2 + G(x), \tag{10}$$

with

$$G(x) = G_\mu(x) := \int_0^x g(s)\, ds = \frac{1}{2}\big(b(x^+)^2 + a(x^-)^2\big) - \mu x.$$

System (E_μ) has a unique equilibrium point

$$P = P_\mu = (\bar{x}, 0),$$

with

$$\bar{x} = \bar{x}_\mu := \begin{cases} \frac{\mu}{b} & \text{if } \mu > 0, \\ \frac{\mu}{a} & \text{if } \mu < 0, \end{cases}$$

which is a global center surrounded by periodic trajectories traversed clockwise. The energy $\mathcal{E}^\mu$ achieves its minimum value $\mathcal{E}^\mu_{\min}$ at the point P_μ, with

$$\mathcal{E}^\mu_{\min} = G_\mu(\bar{x}_\mu) = \min_{\mathbb{R}} G_\mu = \begin{cases} -\frac{\mu^2}{2b} & \text{if } \mu > 0, \\ -\frac{\mu^2}{2a} & \text{if } \mu < 0, \end{cases}$$

and, moreover, $\mathcal{E}^\mu(0, 0) = 0$.

For every $e > \mathcal{E}^\mu_{\min}$, the level line

$$\Gamma^\mu(e) := \big\{(x, y) \in \mathbb{R}^2\colon\ \mathcal{E}^\mu(x, y) = e\big\}$$

(which is symmetric with respect to the x-axis) is a simple closed curve surrounding P_μ. This curve is strictly star-shaped with respect to P_μ and intersects the x-axis at two points $H = (h, 0)$ and $K = (k, 0)$ with

$$h = h^\mu(e) < \bar{x}_\mu < k = k^\mu(e), \qquad G_\mu(h) = G_\mu(k) = e.$$

$\Gamma^\mu(e)$ is also a periodic orbit of (E_μ) whose fundamental period will be denoted by $\mathcal{T}^\mu(e)$. An explicit formula for the computation of $\mathcal{T}^\mu(e)$ is given in the next section.

By an *annulus* around P_μ we mean a compact set defined by

$$\mathcal{A}^\mu(e_1, e_2) := \big\{(x, y) \in \mathbb{R}^2\colon\ e_1 \leqslant \mathcal{E}^\mu(x, y) \leqslant e_2\big\}, \quad \text{with } \mathcal{E}^\mu_{\min} < e_1 < e_2.$$

For our main result, the following definition is crucial.

Definition 1.3. Let $\mathcal{A}^r = \mathcal{A}^r(c_1, c_2)$ and $\mathcal{A}^s = \mathcal{A}^s(d_1, d_2)$ be two annuli around P_r and P_s, respectively. We say that $\mathcal{A}^r$ and $\mathcal{A}^s$ are linked if

$$h^s(d_1) \leqslant h^r(c_2) < h^r(c_1) \leqslant k^s(d_1) < k^s(d_2) \leqslant k^r(c_1).$$

We notice that if $\mathcal{A}^r$ and $\mathcal{A}^s$ are two linked annuli, then

$$\mathcal{A}^r \cap \mathcal{A}^s = \mathcal{P} \cup \mathcal{Q},$$

with

$$\mathcal{P} \subseteq \mathbb{R} \times (-\infty, 0], \qquad \mathcal{Q} \subseteq \mathbb{R} \times [0, +\infty)$$

and such that $\mathcal{P} \cap \mathcal{Q}$ is either empty or is a singleton or it consists of two points. In any case, the intersection (if nonempty) is contained in the x-axis.

Rewriting Eq. (4) in the phase-plane

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -(bx^+ - ax^-) + p_{r,s}(t), \end{cases} \tag{11}$$

our main result is the following:

Theorem 1.2. *For any choice of the constants a, b, r, s satisfying* (6) *and* (9) *and for every pair of linked annuli $\mathcal{A}^r = \mathcal{A}^r(c_1, c_2)$ and $\mathcal{A}^s = \mathcal{A}^s(d_1, d_2)$, with $c_1 \geqslant 0$ and $d_1 \geqslant 0$, the following conclusion holds*:

For every integer $m \geqslant 2$ there exist two positive constants α^*, β^*, such that, for each

$$\tau_r > \alpha^* \quad \text{and} \quad \tau_s > \beta^*$$

the Poincaré map ψ associated to system (11) *induces chaotic dynamics on m symbols on $\mathcal{P}$ and on $\mathcal{Q}$.*

As we will see along the proof, the chaotic dynamics according to Definition 1.2 has a precise interpretation in terms of the oscillatory behavior of the solutions of Eq. (11). In fact, the sets $\mathcal{K}_i$ (for $i = 1, \ldots, m$) are defined by a correspondence with the number of rotations of the solutions around the point P_r. Hence, our solutions oscillate a certain number of times around P_r in the interval $[0, \tau_r[$ and then they oscillate around P_s in the interval $[\tau_r, \tau_r + \tau_s[$. To any sequence $(s_i)_{i \in \mathbb{Z}}$ on m symbols it will correspond a solution having a prescribed number of oscillations around P_r in the time interval $[iT, iT + \tau_r[$ and, moreover, according to our definition of chaos, for any periodic sequence of m symbols, we can associate a corresponding periodic solution of (11).

Theorem 1.2 is stable with respect to small perturbations. Indeed we have

Theorem 1.3. *For any choice of the constants a, b, r, s satisfying* (6) *and* (9) *and for every pair of linked annuli $\mathcal{A}^r = \mathcal{A}^r(c_1, c_2)$ and $\mathcal{A}^s = \mathcal{A}^s(d_1, d_2)$, with $c_1 \geqslant 0$ and $d_1 \geqslant 0$, it follows that for every integer $m \geqslant 2$ there exist two positive constants α^{**}, β^{**}, such that for each*

$$\tau_r > \alpha^{**} \quad \text{and} \quad \tau_s > \beta^{**},$$

there is $\delta = \delta_{\tau_r, \tau_s} > 0$ such that for any $c \in [-\delta, \delta]$ and any T-periodic forcing term $p(\cdot)$ with

$$\int_0^T \left| p(t) - p_{r,s}(t) \right| dt < \delta, \quad T = \tau_r + \tau_s,$$

the same conclusion of Theorem 1.2 *holds with respect to the Poincaré map ϕ associated to the equation*

$$u'' + cu' + bu^+ - au^- = p(t). \tag{12}$$

Both Theorems 1.2 and 1.3 apply to the Lazer–McKenna suspension bridge model (see also the comments and the references in [29]). Our results are strongly related also to the study of periodically forced asymmetric oscillators (see [24] for a recent survey on this subject).

As a final remark (before proceeding to the proofs), we notice that the assumption about the existence of linked annuli is not restrictive. Indeed, such kind of annuli can always be found whenever $a \neq b$ and $r \neq s$. On the contrary, our hypothesis is explicitly assumed in order to show the possibility of obtaining a large number of regions containing initial points corresponding to chaotic like solutions (see Section 3 for more details).

2. Proof of the main results

For any given choice of the coefficients a, b, r, s satisfying (6) and (9), we consider the equation

$$u'' + bu^+ - au^- = p_{r,s}(t)$$

and study it as the equivalent first order system in the phase-plane

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -(bx^+ - ax^-) + p_{r,s}(t). \end{cases} \tag{13}$$

As a simple consequence of the fundamental theory of ODEs we know that for every initial point $z_0 = (x_0, y_0) \in \mathbb{R}^2$ and every $t_0 \in \mathbb{R}$, there exists a unique solution $\zeta(t) := \zeta(t; t_0, z_0)$ of (13) such that $\zeta(t_0) = z_0$ and the Poincaré map

$$\psi : z_0 \mapsto \zeta(t_0 + T; t_0, z_0) \tag{14}$$

is well defined as a homeomorphism of the plane onto itself. As a first instance, we start at $t_0 = 0$ and decompose ψ as

$$\psi = \psi_s \circ \psi_r, \tag{15}$$

where ψ_r and ψ_s are the Poincaré maps

$$\psi_r : z_0 \mapsto \zeta(\tau_r; 0, z_0), \qquad \psi_s : z_0 \mapsto \zeta(\tau_r + \tau_s; \tau_r, z_0). \tag{16}$$

By the special form of $p_{r,s}$ we have that ψ_r and ψ_s are the motions along the time intervals $[0, \tau_r]$ and $[0, \tau_s]$ associated to the dynamical systems defined by (E_μ) for $\mu = r$ and $\mu = s$, respectively. In particular,

$$\psi_s : z_0 \mapsto \zeta(\tau_s; 0, z_0).$$

In order to apply our topological approach it is useful to evaluate the periods of the orbits of these systems. To this end, arguing as in [29], we compute the time-mapping formula for the solutions of (E_μ) with $\mu \neq 0$. Accordingly, we distinguish the following two cases.

Case (A). We consider the orbits of system (E_μ) with energy

$$\mathcal{E}^\mu_{min} < e \leqslant 0.$$

In this case, as proved in [29], all the orbits have the same fundamental period given by

$$\mathcal{T}^\mu(e) = \begin{cases} \frac{2\pi}{\sqrt{b}} & \text{if } \mu > 0, \\ \frac{2\pi}{\sqrt{a}} & \text{if } \mu < 0. \end{cases} \tag{17}$$

Case (B). Now we consider the orbits of system (E_μ) with energy

$$e > 0.$$

We have two points x^μ_- and x^μ_+, with

$$x^\mu_- < 0 < x^\mu_+,$$

such that $G_\mu(x^\mu_-) = G_\mu(x^\mu_+) = e$. The half period $\mathcal{T}^\mu(e)/2$ is the time needed to move from $(x^\mu_-, 0)$ to $(x^\mu_+, 0)$ in the set $y \geqslant 0$. We split such time in two parts, $\mathcal{T}_-$ and $\mathcal{T}_+$, corresponding to the intervals $[x^\mu_-, 0]$ and $[0, x^\mu_+]$. For the next computations, we suppose, for a moment, that $\mu > 0$.

Using the classical time-mapping formula, for the first interval, we find

$$\mathcal{T}_- = \int_{x^\mu_-}^{0} \frac{1}{\sqrt{2(e - G(x))}}\, dx = \int_{x^\mu_-}^{0} \frac{1}{\sqrt{(2e - ax^2 + 2\mu x)}}\, dx = \frac{1}{\sqrt{a}} \left(\frac{\pi}{2} - \arctan \frac{\mu}{\sqrt{2ea}} \right).$$

For the second interval we obtain

$$\mathcal{T}_+ = \int_{0}^{x^\mu_+} \frac{1}{\sqrt{2(e - G(x))}}\, dx = \int_{0}^{x^\mu_+} \frac{1}{\sqrt{(2e - bx^2 + 2\mu x)}}\, dx = \frac{1}{\sqrt{b}} \left(\frac{\pi}{2} + \arctan \frac{\mu}{\sqrt{2eb}} \right),$$

so that we have

$$\mathcal{T}^\mu(e) = \frac{\pi}{\sqrt{b}} + \frac{\pi}{\sqrt{a}} + \frac{2}{\sqrt{b}} \arctan \frac{\mu}{\sqrt{2eb}} - \frac{2}{\sqrt{a}} \arctan \frac{\mu}{\sqrt{2ea}}. \tag{18}$$

Repeating the computations for $\mu < 0$ we find again the same formula (18) which thus holds for every $\mu \in \mathbb{R} \setminus \{0\}$.[1]

[1] Actually, the formula is true also for $\mu = 0$ (the case of an isochronous center).

From (17) and (18) we easily obtain

$$\frac{d}{de}\mathcal{T}^{\mu}(e) = \begin{cases} 0 & \forall e \in]\mathcal{E}^{\mu}_{\min}, 0], \\ \frac{2\sqrt{2}\mu\sqrt{e}(b-a)}{(2eb+\mu^2)(2ea+\mu^2)} \neq 0 & \forall e \in]0, +\infty), \end{cases}$$

and therefore we see that $\mathcal{T}^{\mu}$ is a monotone function of the energy. In particular, $e \mapsto \mathcal{T}^{\mu}(e)$ is strictly increasing on $[0, +\infty)$ for $\mu(b-a) > 0$ and strictly decreasing on $[0, +\infty)$ for $\mu(b-a) < 0$. Moreover,

$$\mathcal{T}^{\mu}(e) \to \frac{\pi}{\sqrt{b}} + \frac{\pi}{\sqrt{a}}, \quad \text{as } e \to +\infty.$$

After these preliminary computations on the time-maps, we pass now to the verification of assumptions (H_r) and (H_s) of Theorem 1.1.

In view of the above time-mapping estimates and in order to avoid the need to distinguish between various different cases, from now on, we assume

$$0 < a < b$$

and omit to investigate the case of a reverse inequality which can be treated in the same manner.

Arguing like in [29], we introduce a system of polar coordinates (θ, ρ) whose origin is placed at $P_r = (\bar{x}_r, 0)$. By expressing the solution $\zeta(\cdot) = \zeta(\cdot; 0, z)$ of (E_r) with $\zeta(0) = z \in \mathcal{P}$, as

$$\zeta(t) = \|\zeta(t) - P_r\| \cdot \big(\cos\big(\theta(t, z)\big), \sin\big(\theta(t, z)\big)\big),$$

we find that, for every $t \in [0, \tau_r]$ and $z \in \mathcal{P}$, *the rotation number for z along* $[0, t]$.

$$\operatorname{rot}(t, z) := \frac{\theta(0, z) - \theta(t, z)}{2\pi}$$

is well defined. The number $2\pi \operatorname{rot}(t, z)$ is the angular displacement along the time interval $[0, t] \subseteq [0, \tau_r]$ for the solution of (E_r) starting at z. Since the definition is chosen in order to count positive the turns around the origin P_r in the clockwise sense, it is easy to prove that the map $t \mapsto \operatorname{rot}(t, z)$ is increasing on $[0, \tau_r]$ for any fixed z. The solutions of the autonomous system (E_r) move along the energy level lines of $\mathcal{E}^r$ and complete exactly one turn at the time $t = \mathcal{T}^r(e)$, where $\mathcal{E}^r(z) = e$. Therefore,

$$\operatorname{rot}(t, z) \begin{cases} < \ell & \text{if and only if} \quad t < \ell\mathcal{T}^r(e), \\ = \ell & \text{if and only if} \quad t = \ell\mathcal{T}^r(e), \\ > \ell & \text{if and only if} \quad t > \ell\mathcal{T}^r(e) \end{cases} \tag{19}$$

holds for every $\ell \in \mathbb{N}$, $\ell \geqslant 1$, provided that $t \leqslant \tau_r$.

By the strict monotonicity of the period map on $[0, +\infty)$, we have $\mathcal{T}^r(c_1) \neq \mathcal{T}^r(c_2)$, where $c_1 < c_2$ are the energy levels of the inner and the outer boundaries of the annulus $\mathcal{A}^r(c_1, c_2)$. Just to fix a case for our discussion, we also assume

$$r > 0$$

so that the map $\mathcal{T}^r(\cdot)$ is strictly increasing on $[0, +\infty)$ and hence

$$\mathcal{T}^r(c_1) < \mathcal{T}^r(c_2).$$

If $r < 0$, the foregoing argument can be repeated verbatim (just interchanging the role of c_1 and c_2).

Now we are going to prove (H_r) of Theorem 1.1 as a consequence of this last inequality. Condition (H_s) will follow from a similar inequality for $\mathcal{T}^s$. To this end, we introduce two oriented rectangles corresponding to the sets $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{N}}$ of Theorem 1.1.

Given a pair of linked annuli $\mathcal{A}^r = \mathcal{A}^r(c_1, c_2)$ and $\mathcal{A}^s = \mathcal{A}^s(d_1, d_2)$, with $c_1 \geqslant 0$ and $d_1 \geqslant 0$, as in the statement of Theorem 1.2, we focus our attention on the sets $\mathcal{P}$ and $\mathcal{Q}$. We recall that, with reference to Fig. 2, the sets $\mathcal{P}$ and $\mathcal{Q}$ are, respectively, the lower and the upper intersections of the two annuli. On the boundary of $\mathcal{P}$ we select the two arcs

$$\mathcal{P}_1^- := \mathcal{P} \cap \big\{(x, y):\ \mathcal{E}^r(x, y) = c_1\big\}, \qquad \mathcal{P}_2^- := \mathcal{P} \cap \big\{(x, y):\ \mathcal{E}^r(x, y) = c_2\big\},$$

and we also set

$$\mathcal{P}^- := \mathcal{P}_1^- \cup \mathcal{P}_2^-, \qquad \widetilde{\mathcal{P}} := (\mathcal{P}, \mathcal{P}^-).$$

In a similar manner, we define

$$\mathcal{Q}_1^- := \mathcal{Q} \cap \big\{(x, y):\ \mathcal{E}^s(x, y) = d_1\big\}, \qquad \mathcal{Q}_2^- := \mathcal{Q} \cap \big\{(x, y):\ \mathcal{E}^s(x, y) = d_2\big\},$$

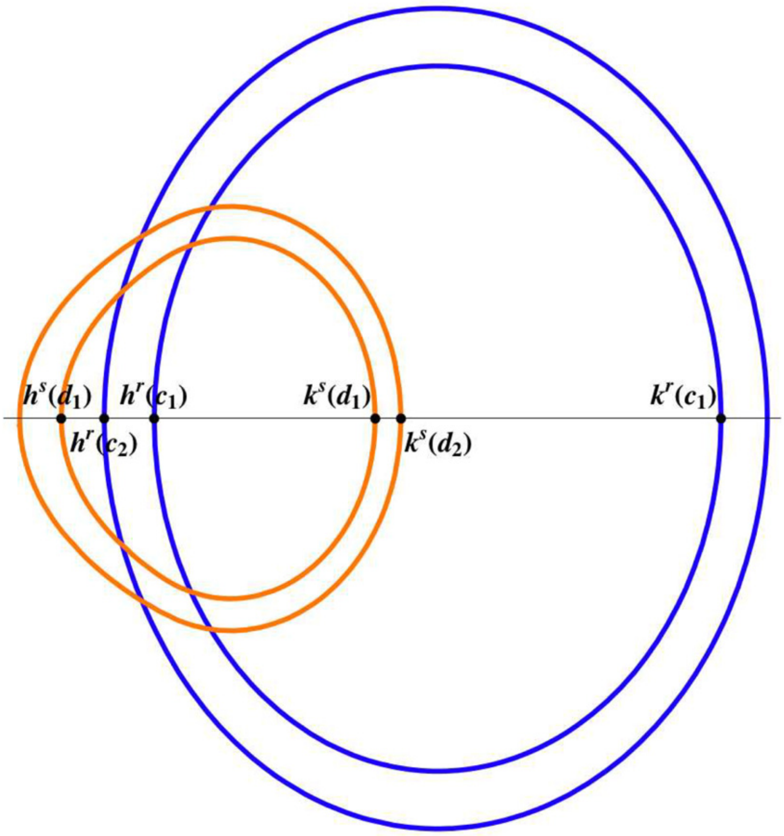

Fig. 2. Example of two linked annuli $\mathcal{A}^r(c_1, c_2)$ (at the right) and $\mathcal{A}^s(d_1, d_2)$ (at the left) according to Definition 1.3.

and

$$\mathcal{Q}^- := \mathcal{Q}_1^- \cup \mathcal{Q}_2^-, \qquad \widetilde{\mathcal{Q}} := (\mathcal{Q}, \mathcal{Q}^-).$$

With these definitions, we can prove the "stretching along the paths" properties.

Let $\gamma = \gamma(\lambda) : [0,1] \to \mathcal{P}$ be a continuous map such that $\gamma(0) \in \mathcal{P}_1^-$ and $\gamma(1) \in \mathcal{P}_2^-$. For each $\lambda \in [0,1]$ we consider the solution of (E_r) having its initial point in $\gamma(\lambda)$ in order to define the map

$$[0,1] \ni \lambda \mapsto \psi_r\big(\gamma(\lambda)\big) = \zeta\big(\tau_r; 0, \gamma(\lambda)\big),$$

as well as its associated angular coordinate

$$[0,1] \ni \lambda \mapsto \theta\big(\tau_r, \gamma(\lambda)\big).$$

The initial point $z = \gamma(\lambda) \in \mathcal{P}$ belongs to the half-plane $y \leqslant 0$, hence we assume

$$\theta\big(0, \gamma(\lambda)\big) \in [-\pi, 0], \quad \forall \lambda \in [0,1]. \tag{20}$$

By (19) we have

$$\operatorname{rot}\big(\tau_r, \gamma(0)\big) \geqslant \left\lfloor \frac{\tau_r}{\mathcal{T}^r(c_1)} \right\rfloor \quad \text{and} \quad \operatorname{rot}\big(\tau_r, \gamma(1)\big) \leqslant \left\lceil \frac{\tau_r}{\mathcal{T}^r(c_2)} \right\rceil$$

and

$$\Delta := \operatorname{rot}\big(\tau_r, \gamma(0)\big) - \operatorname{rot}\big(\tau_r, \gamma(1)\big) \geqslant \left\lfloor \frac{\tau_r}{\mathcal{T}^r(c_1)} \right\rfloor - \left\lceil \frac{\tau_r}{\mathcal{T}^r(c_2)} \right\rceil > \frac{\mathcal{T}^r(c_2) - \mathcal{T}^r(c_1)}{\mathcal{T}^r(c_1)\mathcal{T}^r(c_2)} \tau_r - 2.$$

As a consequence, taking a sufficiently large τ_r, namely

$$\tau_r > \alpha^* := \frac{\mathcal{T}^r(c_1)\mathcal{T}^r(c_2)}{\mathcal{T}^r(c_2) - \mathcal{T}^r(c_1)}(m+4), \tag{21}$$

we have

$$\theta\big(\tau_r,\gamma(1)\big)-\theta\big(\tau_r,\gamma(0)\big)=\theta\big(0,\gamma(1)\big)-\theta\big(0,\gamma(0)\big)+2\pi\Delta>2\pi(m+4-2)-\pi>2\pi(m+1),$$

where (20) has been used to find a lower bound for $\theta(0,\gamma(1))-\theta(0,\gamma(0))$.

We introduce the auxiliary map

$$\omega(\lambda):=\theta\big(\tau_r,\gamma(\lambda)\big),\quad \forall\lambda\in[0,1]$$

which is continuous and has its range contained in $(-\infty,0[$. By the inequality $\omega(1)-\omega(0)>2\pi(m+1)$, we know that $\omega([0,1])/2\pi$ is an interval whose length is strictly larger than $m+1$. Therefore we conclude that there exist at least $m+1$ consecutive integers contained in the interval $\omega([0,1])/2\pi$. As a consequence there exists an integer $k\leqslant -1$ such that

$$\left\{\frac{\omega(\lambda)}{2\pi}:\ \lambda\in[0,1]\right\}\supseteq[k-m,k]=\bigcup_{i=1}^{m}[k-i,k-i+1]\supseteq\bigcup_{i=1}^{m}\left[k-i,k-i+\frac{1}{2}\right].$$

Recalling the definition of $\omega(\lambda)$ and putting $k=-\ell$, with $\ell\geqslant 1$, we can rewrite the above formula as

$$[\alpha,\beta]:=\big\{\theta\big(\tau_r,\gamma(\lambda)\big):\ \lambda\in[0,1]\big\}\supseteq\bigcup_{i=1}^{m}\big[2\pi(-\ell-i),2\pi(-\ell-i)+\pi\big].$$

Since the interval $[\alpha,\beta]$ contains m subintervals of the form $[2j\pi,2j\pi+\pi]$, by the Bolzano theorem we can find m pairwise disjoint maximal subintervals $[\lambda_i',\lambda_i'']$ of $[0,1]$ such that

$$\theta\big(\tau_r,\gamma(\lambda)\big)\in\big]-2\pi(\ell+i),-2\pi(\ell+i)+\pi\big[,\quad \forall\lambda\in\,]\lambda_i',\lambda_i''[,\ \text{for } i=1,\dots,m.$$

Moreover, we have

$$\theta\big(\tau_r,\gamma(\lambda_i')\big)=-2\pi(\ell+i),\qquad \theta\big(\tau_r,\gamma(\lambda_i'')\big)=-2\pi(\ell+i)+\pi.$$

Now we introduce the m nonempty and pairwise disjoint compact sets

$$\mathcal{K}_i:=\big\{P_0\in\mathcal{P}:\ \theta(\tau_r,P_0)\in\big[-2\pi(\ell+i),-2\pi(\ell+i)+\pi\big]\big\},\tag{22}$$

for $i=1,\dots,m$, and claim that

$$(\mathcal{K}_i,\psi_r):\widetilde{\mathcal{P}}\Leftrightarrow\!\!\!\to\widetilde{\mathcal{Q}},\quad \forall i=1,\dots,m,\tag{23}$$

where ψ_r is the Poincaré map defined in the first part of (16). Let us fix an index $i\in\{1,\dots,m\}$ and observe that, by the definition of $\mathcal{K}_i$ and the properties of $\theta(\tau_r,\gamma(\lambda))$ that we have just proved, it holds that:

$$\gamma(\lambda)\in\mathcal{K}_i,\qquad \psi_r\big(\gamma(\lambda)\big)\in\mathcal{A}^r(c_1,c_2)\cap\big\{(x,y)\in\mathbb{R}^2:\ y\geqslant 0\big\},\quad \forall\lambda\in[\lambda_i',\lambda_i'']$$

and

$$\psi_r\big(\gamma(\lambda_i'')\big)\in(-\infty,\bar{x}_r[\,\times\{0\},\qquad \psi_r\big(\gamma(\lambda_i')\big)\in\,]\bar{x}_r,+\infty)\times\{0\}.$$

Since

$$\mathcal{E}^s\big(\psi_r\big(\gamma(\lambda_i'')\big)\big)\leqslant d_1\quad\text{and}\quad \mathcal{E}^s\big(\psi_r\big(\gamma(\lambda_i')\big)\big)\geqslant d_2,$$

there exists a subinterval $[t_i',t_i'']\subseteq[\lambda_i',\lambda_i'']$ such that

$$\psi_r\big(\gamma(\lambda)\big)\in\mathcal{Q},\quad \forall\lambda\in[t_i',t_i''],$$

as well as

$$\mathcal{E}^s\big(\psi_r\big(\gamma(t_i'')\big)\big)=d_1\quad\text{and}\quad \mathcal{E}^s\big(\psi_r\big(\gamma(t_i')\big)\big)=d_2,$$

that is

$$\psi_r\big(\gamma(t_i')\big)\in\mathcal{Q}_1^-\quad\text{and}\quad \psi_r\big(\gamma(t_i'')\big)\in\mathcal{Q}_2^-.$$

This proves (23) which is precisely (H_r) of Theorem 1.1 for the oriented rectangle $\widetilde{\mathcal{M}}:=\widetilde{\mathcal{P}}$.

The proof of (H_s) for $\widetilde{\mathcal{N}}:=\widetilde{\mathcal{Q}}$ is much simpler, because we only need to repeat the above argument with respect the flow associated to system (E_s) and the annulus $\mathcal{A}^s(d_1,d_2)$, by taking as a starting set $\mathcal{Q}$ (instead of $\mathcal{P}$) and $m=1$. In such a case a lower estimate for τ_s can be given by

$$\tau_s > \beta^* := \frac{5\mathcal{T}^s(d_1)\mathcal{T}^s(d_2)}{|\mathcal{T}^s(d_2) - \mathcal{T}^s(d_1)|}. \tag{24}$$

Having proved (H_r) and (H_s) we can apply Theorem 1.1 to the Poincaré map ψ for system (13) defined in (15), thus obtaining the existence of chaotic dynamics on m symbols on a compact subset of $\mathcal{P}$.

The same argument applies also for the proof of chaotic dynamics in the set $\mathcal{Q}$. We have just to define appropriately a new orientation for the sets $\mathcal{M} := \mathcal{Q}$ and $\mathcal{N} := \mathcal{P}$. More precisely, we define

$$\mathcal{M}_1^- := \mathcal{Q} \cap \{(x, y):\ \mathcal{E}^r(x, y) = c_1\}, \qquad \mathcal{M}_2^- := \mathcal{Q} \cap \{(x, y):\ \mathcal{E}^r(x, y) = c_2\},$$

and

$$\mathcal{N}_1^- := \mathcal{P} \cap \{(x, y):\ \mathcal{E}^s(x, y) = d_1\}, \qquad \mathcal{N}_2^- := \mathcal{P} \cap \{(x, y):\ \mathcal{E}^s(x, y) = d_2\}.$$

In order to prove (H_r) we start with a continuous path $\gamma : [0, 1] \to \mathcal{Q}$ with $\mathcal{E}^r(\gamma(0)) = c_1$ and $\mathcal{E}^r(\gamma(1)) = c_2$ and repeat (with obvious changes) the previous proof. The estimates for the angular coordinate are of the same kind, assuming, instead of (20), that

$$\theta(0, \gamma(\lambda)) \in [0, \pi], \quad \forall \lambda \in [0, 1],$$

since $\mathcal{Q}$ is contained in the second quadrant (relatively to the system with origin in P_r). Similarly, we prove (H_s).

This concludes the proof of Theorem 1.2.

We sketch now some details for the proof of Theorem 1.3. For brevity, we confine ourselves only to the task of verifying the presence of chaotic dynamics in the set $\mathcal{P}$ and thus following the proof of the first part of Theorem 1.2. Actually, the proof of Theorem 1.3 follows from the observation that our geometric framework is stable with respect to small perturbations and therefore the previous result applies to the system

$$\begin{cases} \dot{x} = y + f_1(t, x, y), \\ \dot{y} = -(bx^+ - ax^-) + p_{r,s}(t) + f_2(t, x, y), \end{cases} \tag{25}$$

where $f_1, f_2 : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ satisfy the Carathéodory assumptions (see [13]) and are T-periodic in the t-variable ($T = \tau_r + \tau_s$). Clearly, (25) contains Eq. (12) as a particular case. The hypotheses on $p(\cdot)$ and $c \in \mathbb{R}$ in the statement of Theorem 1.3 lead to suitable assumptions of “smallness” on f_1 and f_2 that will be explained in a more detailed form in the course of the proof.

First of all, we suppose that solutions of (25) are uniquely determined by their initial conditions and are globally defined (at least forward in time). Such an assumption is satisfied for Eq. (12). In this manner, for each $z_0 \in \mathbb{R}^2$ and $t_0 \in \mathbb{R}$, there is a unique solution $\xi(\cdot; t_0, z_0)$ of (25) which is defined for all $t \geqslant t_0$. We split the Poincaré map

$$\phi : z_0 \to \xi(T; 0, z_0)$$

as the composition of the two maps

$$\phi_r : z_0 \to \xi(\tau_r; 0, z_0), \qquad \phi_s : w_0 \to \xi(T; \tau_r, w_0).$$

As a next step, we define the oriented rectangles $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{N}}$.

Besides the sets

$$\mathcal{P}_1^- := \mathcal{P} \cap \{(x, y):\ \mathcal{E}^r(x, y) = c_1\}, \qquad \mathcal{P}_2^- := \mathcal{P} \cap \{(x, y):\ \mathcal{E}^r(x, y) = c_2\},$$

and

$$\mathcal{Q}_1^- := \mathcal{Q} \cap \{(x, y):\ \mathcal{E}^s(x, y) = d_1\}, \qquad \mathcal{Q}_2^- := \mathcal{Q} \cap \{(x, y):\ \mathcal{E}^s(x, y) = d_2\},$$

considered in the first part of the proof of Theorem 1.2, we introduce the sets

$$\begin{aligned} &\mathcal{M} := \{(x, y) \in \mathcal{P}:\ c_1' \leqslant \mathcal{E}^r(x, y) \leqslant c_2'\}, \\ &\mathcal{M}_1^- := \mathcal{M} \cap \{(x, y):\ \mathcal{E}^r(x, y) = c_1'\}, \qquad \mathcal{M}_2^- := \mathcal{M} \cap \{(x, y):\ \mathcal{E}^r(x, y) = c_2'\}, \\ &\mathcal{N} := \{(x, y) \in \mathcal{Q}:\ d_1' \leqslant \mathcal{E}^s(x, y) \leqslant d_2'\}, \\ &\mathcal{N}_1^- := \mathcal{N} \cap \{(x, y):\ \mathcal{E}^s(x, y) = d_1'\}, \qquad \mathcal{N}_2^- := \mathcal{N} \cap \{(x, y):\ \mathcal{E}^s(x, y) = d_2'\}, \end{aligned}$$

with

$$c_1 < c_1' < c_2' < c_2, \qquad d_1 < d_1' < d_2' < d_2$$

and get the corresponding oriented rectangles $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{N}}$, defined in the usual manner.

Let $\gamma = \gamma(\lambda) : [0,1] \to \mathcal{M}$ be a continuous map such that $\gamma(0) \in \mathcal{M}_1^-$ and $\gamma(1) \in \mathcal{M}_2^-$. For each $\lambda \in [0,1]$ we consider the solution of (25) having its initial point in $\gamma(\lambda)$ in order to define the map

$$[0,1] \ni \lambda \mapsto \phi_r\big(\gamma(\lambda)\big) = \xi\big(\tau_r; 0, \gamma(\lambda)\big)$$

which may be viewed as a perturbation of the map

$$[0,1] \ni \lambda \mapsto \psi_r\big(\gamma(\lambda)\big) = \zeta\big(\tau_r; 0, \gamma(\lambda)\big)$$

associated to system (11) and already studied in the proof of Theorem 1.2.

Fixing a switching time τ_r such that

$$\tau_r > \alpha^{**} := \frac{\mathcal{T}^r(c_1')\mathcal{T}^r(c_2')}{\mathcal{T}^r(c_2') - \mathcal{T}^r(c_1')}(m+4),$$

we obtain the desired twist property for the angular coordinate associated to the solutions of (11) for $t \in [0, \tau_r]$ and therefore, in the same manner as we proved (H_r) in Theorem 1.2, we have:

there exists $m \geqslant 2$ pairwise disjoint compact sets $\mathcal{K}_1, \dots, \mathcal{K}_m \subseteq \mathcal{M}$ such that

$$(\mathcal{K}_i, \psi_r) : \widetilde{\mathcal{M}} \Longrightarrow \widetilde{\mathcal{Q}}, \quad \forall i = 1, \dots, m.$$

Knowing that for the unperturbed system (11) we have

$$\zeta\big(t; 0, \gamma(\lambda)\big) \in \mathcal{A}^r(c_1', c_2'), \quad \forall t \in \mathbb{R}, \ \forall \lambda \in [0,1],$$

as a consequence of the theorem on the continuous dependence of the solutions from the vector field determining the differential system (see [13]), it follows that if f_1 and f_2 are sufficiently small in the L^1-norm on $[0, \tau_r]$, then

$$\xi\big(t; 0, \gamma(\lambda)\big) \in \mathcal{A}^r(c_1, c_2), \quad \forall t \in [0, \tau_r], \ \forall \lambda \in [0,1],$$

and also

there exists $m \geqslant 2$ pairwise disjoint compact sets $\mathcal{K}_1', \dots, \mathcal{K}_m' \subseteq \mathcal{M}$ such that

$$(\mathcal{K}_i', \phi_r) : \widetilde{\mathcal{M}} \Longrightarrow \widetilde{\mathcal{N}}, \quad \forall i = 1, \dots, m$$

(the sets $\mathcal{K}_i'$ being defined by means of the angular function, as in (22)). The required smallness for f_1 and f_2 can be expressed more precisely as follows:

there exists $\varepsilon_1 > 0$, such that if

$$\big|f_i(t,x,y)\big| \leqslant \varrho_1(t), \quad \forall (x,y) \in \mathcal{A}^r(c_1, c_2), \ \textit{and for a.e. } t \in [0, \tau_r] \ (i = 1, 2),$$

and

$$\int_0^{\tau_r} \varrho_1(t)\, dt < \varepsilon_1,$$

then the above stated properties for $\xi(t; 0, \gamma(\lambda))$ (with $t \in [0, \tau_r]$) and for ϕ_r hold.

The same argument applies if we consider a path $\gamma = \gamma(\lambda) : [0,1] \to \mathcal{N}$ such that $\gamma(0) \in \mathcal{N}_1^-$ and $\gamma(1) \in \mathcal{N}_2^-$. Namely, it follows that if we fix

$$\tau_s > \beta^{**} := \frac{5\mathcal{T}^s(d_1')\mathcal{T}^s(d_2')}{|\mathcal{T}^s(d_2') - \mathcal{T}^s(d_1')|},$$

and if f_1 and f_2 are sufficiently small in the L^1-norm on $[\tau_r, \tau_r + \tau_s]$, then

$$\xi\big(t; \tau_r, \gamma(\lambda)\big) \in \mathcal{A}^s(d_1, d_2), \quad \forall t \in [\tau_r, \tau_r + \tau_s], \ \forall \lambda \in [0,1]$$

and also

$$\phi_s : \widetilde{\mathcal{N}} \Longrightarrow \widetilde{\mathcal{M}}.$$

The smallness required for f_1 and f_2 (in this part of the proof) can be precisely expressed as follows:

there exists $\varepsilon_2 > 0$, *such that if*

$$\left|f_i(t,x,y)\right| \leqslant \varrho_2(t), \quad \forall (x,y) \in \mathcal{A}^s(d_1,d_2), \text{ and for a.e. } t \in [\tau_r, T] \ (i = 1,2),$$

and

$$\int_{\tau_r}^{\tau_r+\tau_s} \varrho_2(t)\,dt < \varepsilon_2,$$

then the above stated properties for $\xi(t;\tau_r,\gamma(\lambda))$ *(with* $t \in [\tau_r, T]$*) and* ϕ_s *hold.*

In conclusion, Theorem 1.1 implies the existence of chaotic dynamics on m symbols induced by the Poincaré map $\phi = \phi_s \circ \phi_r$ of system (25) provided that $|f_1(\cdot,x,y)|$ and $|f_2(\cdot,x,y)|$ (with $(x,y) \in \mathcal{A}^r(c_1,c_2) \cup \mathcal{A}^s(d_1,d_2)$) are bounded by a measurable function which is sufficiently small in the L^1-norm on $[0,T]$.

Remark 2.1. As mentioned in the Introduction, in a recent paper [29] we proved a version of Theorem 1.2 and Theorem 1.3 for (11) for the case

$$a = 0 < b, \qquad s < 0 < r.$$

The same argument we used in the above proof applies (without any significant change) to the situation when

$$a = 0 < b, \qquad 0 < s < r.$$

In this way Theorem 1.2 and Theorem 1.3 complement the results of [29].

Remark 2.2. The underlying geometrical structure related to the proof of our theorems is that of the so-called *Linked Twist Maps* (LTMs). Studies about such kind of mappings have been pursued by several authors both from the theoretical and the applied point of view (see, for instance [10,32] and the references in [30]). Configurations associated to LTMs were discussed by Alekseev in the analysis of the Sitnikov equation for the restricted three body problem [4, pp. 231–237] and [25, pp. 90–94]. We refer also to [30] for more details connecting our approach (Theorem 1.1) to the theory of the linked twist maps [33,34].

Remark 2.3. In the proof of Theorem 1.2, the choice of the constant α^* depends on the integer $m \geqslant 2$, while β^* can be chosen independently of m. This is due to the fact that from the choice of $\tau_r > \alpha^*$ we get (H_r) of Theorem 1.1, while $\tau_s > \beta^*$ is required for the verification of (H_s). Clearly, by taking a large value for β^* we could prove, instead of (H_s), a more general assumption of the form $(\mathcal{K}'_i, \psi_s) : \widetilde{\mathcal{N}} \Longleftrightarrow \widetilde{\mathcal{M}}$, $\forall i = 1,\dots,m'$, with respect to m' pairwise disjoint compact sets $\mathcal{K}'_1,\dots,\mathcal{K}'_{m'} \subseteq \mathcal{N}$. In such a case, we would obtain an even more complicated dynamics on $m \times m'$ symbols.

3. Remarks and related results

This section is devoted to some comments and applications concerning our results of Section 1. For simplicity, we focus our attention mainly to Theorem 1.2. Similar considerations could be derived for Theorem 1.3, too.

3.1. Linked annuli

As a first remark, we discuss the problem of finding two linked annuli, given a, b with $a > 0$, $b > 0$ and $a \neq b$ as well as r, s with $r \neq 0$, $s \neq 0$ and $r > s$.

To this aim, we observe that it will be sufficient to produce two *linked orbits*, with nonempty intersection and then construct an annulus (possibly narrow) around each orbit. To apply Theorem 1.2 we also need the time-map nonconstant in a neighborhood of each orbit. Accordingly, we consider the energy level lines of system (E_μ) with $\mu \in \{r,s\}$ and positive energy.

In order to simplify the procedure, we fix an orbit, say for (E_r), and then show how to find a suitable linked orbit for (E_s).

Let $c > 0$ be an arbitrary constant and consider the equation

$$b(x^+)^2 + a(x^-)^2 - 2rx = 2c \tag{26}$$

whose solutions, called $h^r(c)$ and $k^r(c)$, with

$$h^r(c) < \bar{x}_r < k^r(c) \quad \text{and} \quad h^r(c) < 0 < k^r(c),$$

correspond to the intersections of the level line $\Gamma^r(c)$ with the x-axis. The explicit forms of $h^r(c)$ and $k^r(c)$ are

$$h^r(c) = \frac{r - \sqrt{r^2 + 2ac}}{a}, \qquad k^r(c) = \frac{r + \sqrt{r^2 + 2bc}}{b}.$$

In order to find a level line $\Gamma^s(d)$ of (E_s), which links with $\Gamma^r(c)$, we evaluate the numbers

$$d_1 := \mathcal{E}^s\big(h^r(c), 0\big), \qquad d_2 := \mathcal{E}^s\big(k^r(c), 0\big).$$

Recalling the expression for the energy function $G_\mu(x)$, we obtain

$$d_1 = \frac{1}{2}a\big(h^r(c)\big)^2 - sh^r(c) = c + \frac{(r-s)(r - \sqrt{r^2 + 2ac}\,)}{a},$$
$$d_2 = \frac{1}{2}b\big(k^r(c)\big)^2 - sk^r(c) = c + \frac{(r-s)(r + \sqrt{r^2 + 2bc}\,)}{b},$$

and hence we find

$$d_1 < d_2, \qquad d_2 > 0.$$

Fixing a constant d with

$$0 < d \in \,]d_1, d_2[,$$

the lines $\Gamma^r(c)$ and $\Gamma^s(d)$ intersect exactly at two points $(\hat{x}, \pm\hat{y})$, with $\hat{y} > 0$. In fact, from

$$\begin{cases} y^2 + b(x^+)^2 + a(x^-)^2 - 2rx = 2c, \\ y^2 + b(x^+)^2 + a(x^-)^2 - 2sx = 2d, \end{cases}$$

we obtain

$$\hat{x} = \frac{d-c}{r-s}.$$

Then, we have to solve the equation in y:

$$y^2 = \frac{2(dr - sc)}{r-s} - b(\hat{x}^+)^2 - a(\hat{x}^-)^2. \tag{27}$$

A simple analysis of the function

$$f(\lambda) := \begin{cases} \frac{2(\lambda r - sc)}{r-s} - b(\frac{\lambda - c}{r-s})^2, & \text{for } \lambda \geqslant c, \\ \frac{2(\lambda r - sc)}{r-s} - a(\frac{\lambda - c}{r-s})^2, & \text{for } \lambda \leqslant c \end{cases}$$

shows that

$$f(\lambda) > 0, \quad \text{if and only if} \quad d_1 < \lambda < d_2.$$

This proves the existence of $\hat{y} > 0$ with $\pm\hat{y}$ solutions of (27).

Having proved that $\Gamma^r(c)$ and $\Gamma^s(d)$ intersect exactly at two points, we can construct linked annuli by simply taking two thin annuli around $\Gamma^r(c)$ and $\Gamma^s(d)$.

3.2. Relations with the Dancer–Fučik spectrum

In [8] and [11] Dancer and Fučik started the investigation of eigenvalue problems for asymmetric oscillators, under various boundary conditions. Following Mawhin [24], we define a Fučik eigenvalue for the T-periodic problem ($T > 0$ fixed) as any pair $(a, b) \in \mathbb{R}^2$ such that the autonomous equation

$$u'' + bu^+ - au^- = 0$$

possesses nontrivial T-periodic solutions. It is possible to prove that for the set $\sigma_F(T)$ of *positive Fučik eigenvalues* we have

$$\sigma_F(T) = \bigcup_{n=1}^{\infty} \sigma_F^n(T),$$

with

$$\sigma_F^n(T) := \left\{(a, b) \in \,]0, +\infty)^2\colon\ \frac{\pi}{\sqrt{a}} + \frac{\pi}{\sqrt{b}} = \frac{T}{n}\right\}.$$

Note that a diagonal point (b,b) belongs to $\sigma_F(T)$ if and only if $b=(2\pi/T)^2n^2$ for some positive integer n, and this means that b is a positive eigenvalue of the differential operator $u\mapsto -u''$ with T-periodic boundary conditions.

Assume now that $a>0$ and $b>0$ are fixed, with $a\neq b$, in order to meet the assumptions of our theorems. In this case, we can look for the set of the periods such that

$$(a,b)\in\sigma_F(T).$$

Clearly, the set of the T's satisfying such a relation is given by

$$\mathcal{T}(a,b):=\left(\frac{\pi}{\sqrt{a}}+\frac{\pi}{\sqrt{b}}\right)\times\{1,2,\dots,n,\dots\}.$$

On the other hand, (1.2) guarantees the existence of chaotic dynamics on m symbols provided that

$$T=\tau_r+\tau_s \quad \text{with } \tau_r>\alpha^*,\ \tau_s>\beta^*. \tag{28}$$

Thus we have the possibility of choosing periods T satisfying the latter relation both in $\mathcal{T}(a,b)$ and in its complement with respect to $]0,+\infty)$. This observation shows that, for the particular choice of the forcing terms we are considering, the presence of chaotic like solutions seems to be independent of the resonance with respect to the Dancer–Fučik spectrum. Further investigations in this direction could be of some interest, especially if one takes into account that there are examples of forcing terms $p(t)$ for which Eq. (2) does not possess bounded solutions for (a,b) a positive Fučik eigenvalue (see [9,19]).

3.3. Estimates for the period and the switching times

From (28) and the lower estimates for α^* and β^* obtained in the proof of Theorem 1.2, we could say that our result refers to a "slow chaos" type dynamics, or, in other words, to complex dynamics generated by the presence of slowly oscillating coefficients. Indeed, from (21) and (24) it follows that the smaller the difference between the fundamental periods of the orbits defining the boundaries of the linked annuli, the larger T must be taken. We point out, however, that at least for certain choices of a,b,r,s, a better estimate for τ_s (which is the length of the time interval in which we have to prove the existence of only one crossing, according to (H_s) of Theorem 1.1) can be provided. In any case, even if for some configurations we could optimize the bounds for one of the two switching times, say for τ_s (or for τ_r), nevertheless we will need assumptions for the other time, say τ_r (or, respectively, τ_s), to be large. Finally, we observe that if τ_r and τ_s in the definition of $p_{r,s}(t)$ are fixed (even small) and we have no possibility to tune them, yet we can prove the existence of chaotic dynamics, according to Theorem 1.2 for the equation

$$u''+bu^+-au^-=p_{r,s}(\varepsilon t), \tag{29}$$

for $\varepsilon>0$, sufficiently small. Clearly, in such a case, the period of the forcing term is T/ε which becomes large for small ε. We refer to [3,28] for previous results in this direction (for other equations), obtained by different techniques.

Otherwise, if τ_r and τ_s are fixed and we do not want to modify the period T of the forcing term, we have still the possibility to produce chaotic dynamics if we can change the coefficients and deal with equation

$$v''+\lambda^2\big(bv^+-av^--p_{r,s}(t)\big)=0, \quad \lambda>0. \tag{30}$$

In fact, we can use Theorem 1.2 after a simple change of variables, by observing that Eq. (30) is equivalent to (29) for $\varepsilon=1/\lambda$. Thus, the conditions on "large time" for Eq. (4) can be transferred to conditions on fixed period and large λ for Eq. (30).

Coming back to the assumption of a fixed pair (a,b), we note that the existence of a multiplicity of periodic solutions and the presence of chaotic dynamics for a large T agree with the uniqueness and stability results found in previous articles dealing with asymmetric nonlinearities and the Lazer–McKenna model (see [1,2,19,27]). In particular, we recall that in [2], Alonso and Ortega found conditions for the global asymptotic stability of the solutions of the equation

$$u''+cu'+g(u)=p(t) \tag{31}$$

(which includes (3)). More precisely, in [2] the authors define two functions $A[\cdot]$ and $B[\cdot]$ depending on the friction coefficient c and prove the existence of a bounded solution which is globally asymptotically stable if

$$A[\inf g']\cdot B[\sup g']<1.$$

The stability result in [2] is independent of the period T of the forcing term $p(\cdot)$. This is not in contradiction with Theorem 1.3 because, in case of Eq. (3) and for $c\to 0^+$, we have

$$A[\inf g']\cdot B[\sup g']>1. \tag{32}$$

In [1], Alonso studied Eq. (31) and, assuming (32), found $\tau_1 < \tau_2$ (depending on $\inf g'$, $\sup g'$ and c) such that the uniqueness and the local stability of the T-periodic solutions hold if

$$T \notin n[\tau_1, \tau_2], \quad \forall n \in \mathbb{N}.$$

It is possible to show that the set $]0, +\infty) \setminus \bigcup_{n=1}^{\infty} n[\tau_1, \tau_2]$ is bounded from above. Therefore, for Eq. (3), with sufficiently small c, there is a unique (locally stable) T-periodic solution, according to Alonso theorem, only if $T < T_0$ for some T_0. This is consistent with Theorem 1.3 which guarantees a multiplicity of periodic solutions for T large, that is for T outside the stability intervals found in [1].

3.4. Extensions to more general equations

It was the aim of this paper that of showing an abundance of solutions with a rich and complicated behavior for Eq. (2) (at least for a suitable choice of $p(t)$). Eq. (2) has been widely studied in the literature and, for our point of view, is also interesting as it contains one of the simplest nonlinearities. From a careful reading of the proof, it is clear that the same kind of results like in Theorem 1.2 can be obtained whenever we are able to produce a geometry of linked twist annuli with nonconstant time-maps. In particular, we could extend our results to some equations like

$$\big(\phi_p(u')\big)' + b\phi_p(x^+) - a\phi_p(x^-) = q(t)$$

(where $\phi_p(u) = |u|^{p-2}u$, $p > 1$ is the one-dimensional p-Laplacian) and even for more general classes of differential operators and nonlinearities (at least for some special forms of the forcing term $q(t)$).

References

[1] J.M. Alonso, Optimal intervals of stability of a forced oscillator, Proc. Amer. Math. Soc. 123 (1995) 2031–2040.
[2] J.M. Alonso, R. Ortega, Boundedness and global asymptotic stability of a forced oscillator, Nonlinear Anal. 25 (1995) 297–309.
[3] A. Ambrosetti, M. Berti, Homoclinics and complex dynamics in slowly oscillating systems, Discrete Contin. Dyn. Syst. 4 (1998) 393–403.
[4] V.I. Arnold, V.V. Kozlov, A.I. Neishtadt, Mathematical Aspects of Classical and Celestial Mechanics. Dynamical Systems III, Encyclopaedia Math. Sci., vol. 3, Springer, Berlin, 1993.
[5] K. Burns, H. Weiss, A geometric criterion for positive topological entropy, Comm. Math. Phys. 172 (1995) 95–118.
[6] E.I. Butikov, Square-wave excitation of a linear oscillator, Amer. J. Phys. 72 (2004) 469–476.
[7] A. Capietto, W. Dambrosio, D. Papini, Superlinear indefinite equations on the real line and chaotic dynamics, J. Differential Equations 181 (2002) 419–438.
[8] E.N. Dancer, Boundary-value problems for weakly nonlinear ordinary differential equations, Bull. Austral. Math. Soc. 15 (1976) 321–328.
[9] E.N. Dancer, Proofs of the results of Boundary-value problems for weakly nonlinear ordinary differential equations, 1976, unpublished.
[10] R.L. Devaney, Subshifts of finite type in linked twist mappings, Proc. Amer. Math. Soc. 71 (1978) 334–338.
[11] S. Fučik, Boundary value problems with jumping nonlinearities, Časopis Pěst. Mat. 101 (1976) 69–87.
[12] Z. Galias, P. Zgliczyński, Abundance of homoclinic and heteroclinic orbits and rigorous bounds for the topological entropy for the Hénon map, Nonlinearity 14 (2001) 909–932.
[13] J.K. Hale, Ordinary Differential Equations, second ed., R.E. Krieger Publishing Co., Huntington, NY, 1980.
[14] A. Huppert, B. Blasius, R. Olinky, L. Stone, A model for seasonal phytoplankton blooms, J. Theoret. Biol. 236 (2005) 276–290.
[15] J. Kennedy, S. Koçak, Y.A. Yorke, A chaos lemma, Amer. Math. Monthly 108 (2001) 411–423.
[16] J. Kennedy, J.A. Yorke, Topological horseshoes, Trans. Amer. Math. Soc. 353 (2001) 2513–2530.
[17] U. Kirchgraber, D. Stoffer, On the definition of chaos, ZAMM Z. Angew. Math. Mech. 69 (1989) 175–185.
[18] A.C. Lazer, P.J. McKenna, Large scale oscillatory behaviour in loaded asymmetric systems, Ann. Inst. H. Poincaré Anal. Non Lineaire 4 (1987) 244–274.
[19] A.C. Lazer, P.J. McKenna, Existence, uniqueness, and stability of oscillations in differential equations with asymmetric nonlinearities, Trans. Amer. Math. Soc. 315 (1989) 721–739.
[20] A.C. Lazer, P.J. McKenna, Large-amplitude periodic oscillations in suspension bridges: Some new connections with nonlinear analysis, SIAM Rev. 32 (1990) 537–578.
[21] M. Levi, On the Littlewood's counterexample of unbounded motions in superquadratic potentials, in: Dynamics Reported: Expositions in Dynamical Systems, vol. 1, Springer, Berlin, 1992, pp. 113–124.
[22] J.E. Littlewood, Unbounded solutions of an equation $y'' + g(y) = p(t)$, with $p(t)$ periodic and bounded and $g(y)/y \to \infty$ as $y \to \pm\infty$, J. Lond. Math. Soc. 41 (1966) 497–507.
[23] P.J. McKenna, Large torsional oscillations in suspension bridges revisited: Fixing an old approximation, Amer. Math. Monthly 106 (1999) 1–18.
[24] J. Mawhin, J. Resonance and nonlinearity: A survey, Ukrainian Math. J. 59 (2007) 197–214.
[25] J. Moser, Stable and Random Motions in Dynamical Systems. With Special Emphasis on Celestial Mechanics, Hermann Weyl Lectures, the Institute for Advanced Study, Ann. of Math. Stud., vol. 77, Princeton University Press, Princeton, NJ, 1973.
[26] R. Olinky, A. Huppert, L. Stone, Seasonal dynamics and thresholds governing recurrent epidemics, J. Math. Biol. 56 (2008) 827–839.
[27] R. Ortega, The first interval of stability of a periodic equation of Duffing type, Proc. Amer. Math. Soc. 115 (1992) 1061–1067.
[28] A.V. Osipov, V.A. Pliss, Relaxation oscillations in a system of Duffing type, Differ. Uravn. 25 (1989) 435–446, 548 (in Russian); translation in Differ. Equ. 25 (1989) 300–308.
[29] A. Pascoletti, F. Zanolin, Example of a suspension bridge ODE model exhibiting chaotic dynamics: A topological approach, J. Math. Anal. Appl. 339 (2008) 1179–1198.
[30] A. Pascoletti, M. Pireddu, F. Zanolin, Multiple periodic solutions and complex dynamics for second order ODEs via linked twist maps, in: Proceedings of the 8th Colloquium on the Qualitative Theory of Differential Equations, Szeged, 2007, Electron. J. Qual. Theory Differ. Equ. 14 (2008) 1–32.
[31] R. Srzednicki, K. Wójcik, A geometric method for detecting chaotic dynamics, J. Differential Equations 135 (1997) 66–82.
[32] R. Sturman, J. Ottino, S. Wiggins, The Mathematical Foundations of Mixing, Cambridge Monogr. Appl. Comput. Math., vol. 22, Cambridge University Press, Cambridge, 2006.

[33] S. Wiggins, Chaos in the dynamics generated by sequence of maps, with application to chaotic advection in flows with aperiodic time dependence, Z. Angew. Math. Phys. 50 (1999) 585–616.
[34] S. Wiggins, J.M. Ottino, Foundations of chaotic mixing, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 362 (2004) 937–970.
[35] K. Wójcik, P. Zgliczyński, Isolating segments, fixed point index, and symbolic dynamics, J. Differential Equations 161 (2000) 245–288.
[36] P. Zgliczyński, M. Gidea, Covering relations for multidimensional dynamical systems, J. Differential Equations 202 (2004) 32–58.
[37] V. Zharnitsky, Breakdown of stability of motion in superquadratic potentials, Comm. Math. Phys. 189 (1997) 165–204.